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Branching Random Walks in Time Inhomogeneous Environments

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Abstract

We study the maximal displacement of branching random walks in a class of time inhomogeneous environments. Specifically, binary branching random walks with Gaussian increments will be considered, where the variances of the increments change over time macroscopically. We find the asymptotics of the maximum up to an $O_P(1)$ (stochastically bounded) error, and focus on the following phenomena: the profile of the variance matters, both to the leading (velocity) term and to the logarithmic correction term, and the latter exhibits a phase transition.

1 Introduction

Branching random walks and their maxima have been studied mostly in space-time homogeneous environments (deterministic or random). For work on the deterministic homogeneous case of relevance to our study we refer to [4] and the recent [1] and [2]. For the random environment case, a sample of relevant papers is [12, 14, 17, 18, 21, 22, 23]. As is well documented in these references, under reasonable hypotheses, in the homogeneous case the maximum grows linearly, with a logarithmic correction, and is tight around its median.

Branching random walks are also studied under some space inhomogeneous environments. A sample of those papers are [3, 7, 9, 13, 15, 16, 19].

Recently, Bramson and Zeitouni [5] and Fang [10] showed that the maxima of branching random walks, recentered around their median, are still tight in time inhomogeneous environments satisfying certain uniform regularity assumptions, in particular, the laws of the increments can vary with respect to time and the walks may have some local dependence. A natural question is to ask, in that situation, what is the asymptotic behavior of the maxima. Similar questions were discussed in the context of branching Brownian motion using

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PDE techniques, see e.g. Nolen and Ryzhik [24], using the fact that the distributions of the maxima satisfy the KPP equation whose solution exhibits a traveling wave phenomenon.

In all these models, while the linear traveling speed of the maxima is a relatively easy consequence of the large deviation principle, the evaluation of the second order correction term, like the ones in Bramson [4] and Addario-Berry and Reed [1], is more involved and requires a detailed analysis of the walks; to our knowledge, it has so far only been performed in the time homogeneous case.

Our goal is to start exploring the time inhomogeneous setup. As we will detail below, the situation, even in the simplest setting, is complex and, for example, the order in which inhomogeneity presents itself matters, both in the leading term and in the correction term.

In this paper, in order to best describe the phenomenon discussed above, we focus on the simplest case of binary branching random walks where the diffusivity of the particles takes two distinct values as a function of time.

We now describe the setup in detail. For $\sigma > 0$, let $N(0,\sigma^2)$ denote the normal distributions with mean zero and variance σ^2. Let n be an integer, and let $\sigma_1^2, \sigma_2^2 > 0$ be given. We start the system with one particle at location 0 at time 0. Suppose that v is a particle at location S_v at time k. Then v dies at time $k+1$ and gives birth to two particles $v1$ and $v2$, and each of the two offspring ($\{vi, i = 1, 2\}$) moves independently to a new location S_{vi} with the increment $S_{vi} - S_v$ independent of S_v and distributed as $N(0,\sigma_1^2)$ if $k < n/2$ and as $N(0,\sigma_2^2)$ if $n/2 \leq k < n$. Let $\mathbb{D}_n$ denote the collection of all particles at time n. For a particle $v \in \mathbb{D}_n$ and $i < n$, we let v^i denote the ith level ancestor of v, that is the unique element of $\mathbb{D}_i$ on the geodesic connecting v and the root. We study the maximal displacement $M_n = \max_{v \in \mathbb{D}_n} S_v$ at time n, for n large. [1]

It will be clear that the analysis extends to a wide class of inhomogeneities with finitely many values and 'macroscopic' change (similar to the description in the previous paragraph), and to the Galton-Watson setup. A universal result that will allow for continuous change of the variances is more complicated, is expected to present different correction terms, and is the subject of further study.

In order to describe the results in a concise way, we recall the notation $O_P(1)$ for stochastically boundedness. That is, if a sequence of random variables R_n satisfies $R_n = O_P(1)$, then, for any $\epsilon > 0$, there exists an M such that $P(|R_n| > M) < \epsilon$ for all n.

An interesting feature of M_n is that the asymptotic behavior depends on the order relation between σ_1^2 and σ_2^2. That is, while

$$M_n = \left(\sqrt{2\log 2}\,\sigma_{\text{eff}}\right) n - \beta \frac{\sigma_{\text{eff}}}{\sqrt{2\log 2}} \log n + O_P(1) \tag{1}$$

is true for some choice of σ_{eff} and β, σ_{eff} and β take different expressions for different ordering of σ_1 and σ_2. Note that (1) is equivalent to say that the sequence $\{M_n - Med(M_n)\}_n$ is tight and

$$Med(M_n) = \left(\sqrt{2\log 2}\,\sigma_{\text{eff}}\right) n - \beta \frac{\sigma_{\text{eff}}}{\sqrt{2\log 2}} \log n + O(1),$$

[1]Since one can understand a branching random walk as a 'competition' between branching and random walk, one may get similar results by fixing the variance and changing the branching rate with respect to time.

where $Med(X) = \sup\{x : P(X \le x) \le \frac{1}{2}\}$ is the median of the random variable X. In the following, we will use superscripts to distinguish different cases, see (2), (3) and (4) below.

A special and well-known case is when $\sigma_1 = \sigma_2 = \sigma$, i.e., all the increments are i.i.d.. In that case, the maximal displacement is described as follows:

$$M_n^{=} = \left(\sqrt{2\log 2}\,\sigma\right) n - \frac{3}{2}\frac{\sigma}{\sqrt{2\log 2}}\log n + O_P(1); \tag{2}$$

the proof can be found in [1], and its analog for branching Brownian motion can be found in [4] using probabilistic techniques and [20] using PDE techniques. This homogeneous case corresponds to (1) with $\sigma_{\text{eff}} = \sigma$ and $\beta = \frac{3}{2}$. In this paper, we deal with the extension to the inhomogeneous case. The main results are the following two theorems.

Theorem 1. *When $\sigma_1^2 < \sigma_2^2$ (increasing variances), the maximal displacement is*

$$M_n^{\uparrow} = \left(\sqrt{(\sigma_1^2+\sigma_2^2)\log 2}\right) n - \frac{\sqrt{\sigma_1^2+\sigma_2^2}}{4\sqrt{\log 2}}\log n + O_P(1), \tag{3}$$

which is of the form (1) *with* $\sigma_{\text{eff}} = \sqrt{\frac{\sigma_1^2+\sigma_2^2}{2}}$ *and* $\beta = \frac{1}{2}$.

Theorem 2. *When $\sigma_1^2 > \sigma_2^2$ (decreasing variances), the maximal displacement is*

$$M_n^{\downarrow} = \frac{\sqrt{2\log 2}(\sigma_1+\sigma_2)}{2} n - \frac{3(\sigma_1+\sigma_2)}{2\sqrt{2\log 2}}\log n + O_P(1), \tag{4}$$

which is of the form (1) *with* $\sigma_{\text{eff}} = \frac{\sigma_1+\sigma_2}{2}$ *and* $\beta = 3$.

For comparison purpose, it is useful to introduce the model of 2^n independent (inhomogeneous) random walks with centered independent Gaussian variables, with variance profile as above. Denote by M_n^{ind} the maximal displacement at time n in this model. Because of the complete independence, it can be easily shown that

$$M_n^{\text{ind}} = \left(\sqrt{(\sigma_1^2+\sigma_2^2)\log 2}\right) n - \frac{\sqrt{\sigma_1^2+\sigma_2^2}}{4\sqrt{\log 2}}\log n + O_P(1) \tag{5}$$

for all choices of σ_1^2 and σ_2^2. Thus, in this case, $\sigma_{\text{eff}} = \sqrt{(\sigma_1^2+\sigma_2^2)/2}$ and $\beta = 1/2$. Thus, the difference between $M_n^{=}$ and M_n^{ind} when $\sigma_1^2 = \sigma_2^2$ lies in the logarithmic correction. As commented (for branching Brownian motion) in [4], the different correction is due to the intrinsic dependence between particles coming from the branching structure in branching random walks.

Another related quantity is the sub-maximum obtained by a greedy algorithm, which only considers the maximum over all decendents of the maximal particle at time $n/2$. Applying (2), we find that the output of such algorithm is

$$\begin{aligned}
&\left(\sqrt{2\log 2}\sigma_1\frac{n}{2} - \frac{3}{2}\frac{\sigma_1}{\sqrt{2\log 2}}\log\frac{n}{2}\right) + \left(\sqrt{2\log 2}\sigma_2\frac{n}{2} - \frac{3}{2}\frac{\sigma_2}{\sqrt{2\log 2}}\log\frac{n}{2}\right) + O_P(1)\\
&= \frac{\sqrt{2\log 2}(\sigma_1+\sigma_2)}{2} n - \frac{3(\sigma_1+\sigma_2)}{2\sqrt{2\log 2}}\log n + O_P(1).
\end{aligned} \tag{6}$$

Comparing (6) with the theorems, we see that this algorithm yields the maximum up to an $O_P(1)$ error in the case of decreasing variances (compare with (4)) but not in the case of increasing variances (compare with (3)) or of homogeneous increments (compare with (2)).

A few comparisons are now in order.

1. When the variances are increasing, $M_n^{\uparrow}$ is asymptotically (up to $O_P(1)$ error) the same as M_n^{ind}, which is exactly the same as the maximum of independent homogeneous random walks with effective variance $\frac{\sigma_1^2+\sigma_2^2}{2}$.
2. When the variances are decreasing, $M_n^{\downarrow}$ shares the same asymptotic behavior with the sub-maximum (6). In this case, a greedy strategy yields the approximate maximum.
3. With the same set of diffusivity constants $\{\sigma_1^2, \sigma_2^2\}$ but different order, $M_n^{\uparrow}$ is greater than $M_n^{\downarrow}$.
4. While the leading order terms in (2), (3) and (4) are continuous in σ_1 and σ_2 (they coincide upon setting $\sigma_1 = \sigma_2$), the logarithmic corrections exhibit a phase transition phenomenon (they are not the same when we let $\sigma_1 = \sigma_2$).

We will prove Theorem 1 in Section 2 and Theorem 2 in Section 3. Before proving the theorems, we state a tightness result.

Lemma 1. *The sequences* $\{M_n^{\uparrow} - Med(M_n^{\uparrow})\}_n$ *and* $\{M_n^{\downarrow} - Med(M_n^{\downarrow})\}_n$ *are tight.*

This lemma follows from Bramson and Zeitouni [5] or Fang [10]. One can write down a similar recursion for the distribution of M_n to the one in those two papers, except for different subscripts and superscripts. Since the argument there depends only on one step of the recursion, it applies here directly without any change and leads to the tightness result in the lemma.

A note on notation: throughout, we use C to denote a generic positive constant, possibly depending on σ_1 and σ_2, that may change from line to line.

2 Increasing Variances: $\sigma_1^2 < \sigma_2^2$

In this section, we prove Theorem 1. We begin in Subsection 2.1 with a result on the fluctuation of an inhomogeneous random walk. In the short Subsection 2.2 we provide large-deviations based heuristics for our results. While it is not used in the actual proof, it explains the leading term of the maximal displacement and gives hints about the derivation of the logarithmic correction term. The actual proof of Theorem 1 is provided in subsection 2.3.

2.1 Fluctuation of an Inhomogeneous Random Walk

Let

$$S_n = \sum_{i=1}^{n/2} X_i + \sum_{i=n/2+1}^{n} Y_i \tag{7}$$

be an inhomogeneous random walk, where $X_i \sim N(0, \sigma_1^2)$, $Y_i \sim N(0, \sigma_2^2)$, and X_i and Y_i are independent. Define

$$s_{k,n}(x) = \begin{cases} \dfrac{\sigma_1^2 k}{(\sigma_1^2 + \sigma_2^2)\frac{n}{2}} x, & 0 \le k \le \dfrac{n}{2}, \\ \dfrac{\sigma_1^2 \frac{n}{2} + \sigma_2^2 (k - \frac{n}{2})}{(\sigma_1^2 + \sigma_2^2)\frac{n}{2}} x, & \dfrac{n}{2} \le k \le n, \end{cases} \tag{8}$$

and

$$f_{k,n} = \begin{cases} c_f k^{2/3}, & k \le n/2, \\ c_f (n-k)^{2/3}, & n/2 < k \le n. \end{cases} \tag{9}$$

As the following lemma says, conditioned on $\{S_n = x\}$, the path of the walk S_n follows $s_{k,n}(x)$ with fluctuation less than or equal to $f_{k,n}$ at level $k \le n$.

Lemma 2. *There exists a constant $C > 0$ (independent of n) such that*

$$P(S_n(k) \in [s_{k,n}(S_n) - f_{k,n}, s_{k,n}(S_n) + f_{k,n}] \ \text{ for all } \ 0 \le k \le n | S_n) \ge C,$$

where $S_n(k)$ is the sum of the first k summands of S_n, i.e.,

$$S_n(k) = \begin{cases} \displaystyle\sum_{k=1}^{k} X_k, & k \le n/2, \\ \displaystyle\sum_{k=1}^{n/2} X_k + \sum_{k=n/2+1}^{k} Y_k, & n/2 < k \le n. \end{cases}$$

Proof. Let $\tilde{S}_{k,n} = S_n(k) - s_{k,n}(S_n)$. Then, similar to Brownian bridge, one can check that $\tilde{S}_{k,n}$ are independent of S_n. To see this, first note that the covariance between $\tilde{S}_{k,n}$ and S_n is

$$Cov(\tilde{S}_{k,n}, S_n) = E\tilde{S}_{k,n} S_n - E\tilde{S}_{k,n} E S_n = E\tilde{S}_{k,n} S_n,$$

since $ES_n = 0$ and $E\tilde{S}_{k,n} = 0$.

For $k \le n/2$,

$$\tilde{S}_{k,n} = \left(1 - \frac{\sigma_1^2 k}{(\sigma_1^2 + \sigma_2^2)\frac{n}{2}}\right) \sum_{i=1}^{k} X_i - \frac{\sigma_1^2 k}{(\sigma_1^2 + \sigma_2^2)\frac{n}{2}} \sum_{i=k+1}^{n/2} X_i - \frac{\sigma_1^2 k}{(\sigma_1^2 + \sigma_2^2)\frac{n}{2}} \sum_{i=n/2+1}^{n} Y_i.$$

Expand $\tilde{S}_{k,n} S_n$, take expectation, and then all terms vanish except for those containing X_i^2 and Y_i^2. Taking into account that $EX_i^2 = \sigma_1^2$ and $EY_i^2 = \sigma_2^2$, one has

$$\begin{aligned} & Cov(\tilde{S}_{k,n}, S_n) = E\tilde{S}_{k,n} S_n \\ = & \left(1 - \frac{\sigma_1^2 k}{(\sigma_1^2 + \sigma_2^2)\frac{n}{2}}\right) \sum_{i=1}^{k} EX_i^2 - \frac{\sigma_1^2 k}{(\sigma_1^2 + \sigma_2^2)\frac{n}{2}} \sum_{i=k+1}^{n/2} EX_i^2 - \frac{\sigma_1^2 k}{(\sigma_1^2 + \sigma_2^2)\frac{n}{2}} \sum_{i=n/2+1}^{n} EY_i^2 \\ = & \left(1 - \frac{\sigma_1^2 k}{(\sigma_1^2 + \sigma_2^2)\frac{n}{2}}\right) k\sigma_1^2 - \frac{\sigma_1^2 k}{(\sigma_1^2 + \sigma_2^2)\frac{n}{2}} (n/2 - k)\sigma_1^2 - \frac{\sigma_1^2 k}{(\sigma_1^2 + \sigma_2^2)\frac{n}{2}} (n/2)\sigma_2^2 \\ = & \ 0. \end{aligned} \tag{10}$$

For $n/2 < k \le n$, one can calculate $Cov(\tilde{S}_{k,n}, S_n) = 0$ similarly as follows. First,

$$\tilde{S}_{k,n} = \frac{\sigma_2^2(n-k)}{(\sigma_1^2+\sigma_2^2)\frac{n}{2}} \sum_{i=1}^{n/2} X_i + \frac{\sigma_2^2(n-k)}{(\sigma_1^2+\sigma_2^2)\frac{n}{2}} \sum_{i=n/2+1}^{k} Y_i - \left(1 - \frac{\sigma_2^2(n-k)}{(\sigma_1^2+\sigma_2^2)\frac{n}{2}}\right) \sum_{i=k+1}^{n} Y_i.$$

Then, expanding $\tilde{S}_{k,n} S_n$ and taking expectation, one has

$$\begin{aligned}
& Cov(\tilde{S}_{k,n}, S_n) = E\tilde{S}_{k,n}S_n \\
= \; & \frac{\sigma_2^2(n-k)}{(\sigma_1^2+\sigma_2^2)\frac{n}{2}} \sum_{i=1}^{n/2} EX_i^2 + \frac{\sigma_2^2(n-k)}{(\sigma_1^2+\sigma_2^2)\frac{n}{2}} \sum_{i=n/2+1}^{k} EY_i^2 - \left(1 - \frac{\sigma_2^2(n-k)}{(\sigma_1^2+\sigma_2^2)\frac{n}{2}}\right) \sum_{i=k+1}^{n} EY_i^2 \\
= \; & \frac{\sigma_2^2(n-k)}{(\sigma_1^2+\sigma_2^2)\frac{n}{2}}(n/2)\sigma_1^2 + \frac{\sigma_2^2(n-k)}{(\sigma_1^2+\sigma_2^2)\frac{n}{2}}(k-n/2)\sigma_2^2 - \left(1 - \frac{\sigma_2^2(n-k)}{(\sigma_1^2+\sigma_2^2)\frac{n}{2}}\right)(n-k)\sigma_2^2 \\
= \; & 0
\end{aligned}$$

Therefore, $\tilde{S}_{k,n}$ are independent of S_n since they are Gaussian. Using this independence,

$$\begin{aligned}
& P(S_n(k) \in [s_{k,n}(S_n) - f_{k,n}, s_{k,n}(S_n) + f_{k,n}] \text{ for all } 0 \le k \le n | S_n) \\
= \; & P(\tilde{S}_{k,n} \in [-f_{k,n}, f_{k,n}] \text{ for all } 0 \le k \le n | S_n) \\
= \; & P(\tilde{S}_{k,n} \in [-f_{k,n}, f_{k,n}] \text{ for all } 0 \le k \le n).
\end{aligned}$$

By calculation similar to (10), $\tilde{S}_{k,n}$ is a Gaussian sequence with mean zero and variance $k\sigma_1^2 \frac{\left((\sigma_1^2+\sigma_2^2)n - 2\sigma_1^2 k\right)}{(\sigma_1^2+\sigma_2^2)n}$ for $k \le n/2$ and $(n-k)\sigma_2^2 \frac{\left((\sigma_1^2+\sigma_2)n - 2\sigma_2^2(n-k)\right)}{(\sigma_1^2+\sigma_2^2)n}$ for $n/2 < k \le n$. The above quantity is

$$1 - P(|\tilde{S}_{k,n}| > f_{k,n}, \text{ for some } 0 \le k \le n) \ge 1 - \sum_{k=1}^{n} P(|\tilde{S}_{k,n}| > f_{k,n}).$$

Using a standard Gaussian estimate, e.g. [8, Theorem 1.4], the above quantity is at least,

$$1 - \sum_{k=1}^{n} \frac{c_0}{\sqrt{k}} e^{-\frac{f_{k,n}^2}{k} c_1} \ge 1 - 2\sum_{k=1}^{\infty} \frac{c_0}{\sqrt{k}} e^{-c_f^2 c_1 k^{1/3}} := C > 0$$

where c_0, c_1 are constants depending on σ_1 and σ_2, and $C > 0$ can be realized by choosing the constant c_f large. This proves the lemma. □

2.2 Sample Path Large Deviation Heuristics

We explain (without giving a proof) what we expect for the order n term of $M_n\uparrow$, by giving a large deviation argument. The exact proof will be postponed to the next subsection. Consider the same S_n as defined in (7) and a function $\phi(t)$ defined on $[0,1]$ with $\phi(0) = 0$. A sample path large deviation result, see [6, Theorem 5.1.2], tells us that the probability for $S_{\lfloor rn \rfloor}$ to be roughly $\phi(r)n$ for $0 \le r \le s \le 1$ is roughly $e^{-nI_s(\phi)}$, where

$$I_s(\phi) = \int_0^s \Lambda_r^*(\dot{\phi}(r))dr, \tag{11}$$

$\dot\phi(r) = \frac{d}{dr}\phi(r)$, and $\Lambda_r^*(x) = \frac{x^2}{2\sigma_1^2}$, for $0 \le r \le 1/2$, and $\frac{x^2}{2\sigma_1^2}$, for $1/2 < r \le 1$. A first moment argument would yield a necessary condition for a walk that roughly follows the path $\phi(r)n$ to exist among the branching random walks,

$$I_s(\phi) \le s \log 2, \quad \text{for all } \ 0 \le s \le 1. \tag{12}$$

This is equivalent to

$$\begin{cases} \displaystyle\int_0^s \frac{\dot\phi^2(r)}{2\sigma_1^2} dr \le s \log 2, & 0 \le s \le \dfrac{1}{2}, \\ \displaystyle\int_0^{\frac{1}{2}} \frac{\dot\phi^2(r)}{2\sigma_1^2} dr + \int_{\frac{1}{2}}^s \frac{\dot\phi^2(r)}{2\sigma_2^2} dr \le s \log 2, & \dfrac{1}{2} \le s \le 1. \end{cases} \tag{13}$$

Otherwise, if (12) is violated for some s_0, i.e., $I_{s_0}(\phi) > s_0 \log 2$, there will be no path following $\phi(r)n$ to $\phi(s_0)n$, since the expected number of such paths is $2^{sn}e^{-nI_s(\phi)} = e^{-(I_s(\phi) - s\log 2)n}$, which decreases exponentially.

Our goal is then to maximize $\phi(1)$ under the constraints (13). By Jensen's inequality and convexity, one can prove that it is equivalent to maximizing $\phi(1)$ subject to

$$\frac{\phi^2(1/2)}{\sigma_1^2} \le \frac{1}{2}\log 2, \ \frac{\phi^2(1/2)}{\sigma_1^2} + \frac{(\phi(1) - \phi(1/2))^2}{\sigma_2^2} \le \log 2. \tag{14}$$

Note that the above argument does not necessarily require $\sigma_1^2 < \sigma_2^2$.

Under the assumption that $\sigma_1^2 < \sigma_2^2$, we can solve the optimization problem with the optimal curve

$$\phi(s) = \begin{cases} \dfrac{2\sigma_1^2\sqrt{\log 2}}{\sqrt{(\sigma_1^2 + \sigma_2^2)}} s, & 0 \le s \le \dfrac{1}{2}, \\ \dfrac{2\sigma_1^2\sqrt{\log 2}}{\sqrt{(\sigma_1^2 + \sigma_2^2)}} \dfrac{1}{2} + \dfrac{2\sigma_2^2\sqrt{\log 2}}{\sqrt{(\sigma_1^2 + \sigma_2^2)}} (s - \dfrac{1}{2}), & \dfrac{1}{2} \le s \le 1. \end{cases} \tag{15}$$

If we plot this optimal curve and the suboptimal curve leading to (6) as in Figure 1, it is easy to see that the ancestor at time $n/2$ of the actual maximum at time n is not a maximum at time $n/2$, since $\frac{2\sigma_1^2\sqrt{\log 2}}{\sqrt{(\sigma_1^2+\sigma_2^2)}} < \sqrt{2\sigma_1^2 \log 2}$. A further rigorous calculation as in the next subsection shows that, along the optimal curve (15), the branching random walks have an exponential decay of correlation. Thus a fluctuation between $n^{1/2}$ and n that is larger than the typical fluctuation of a random walk is admissible. This is consistent with the naive observation from Figure 1. This kind of behavior also occurs in the independent random walks model, explaining why $M_n^{\uparrow}$ and M_n^{ind} have the same asymptotical expansion up to an $O(1)$ error, see (3) and (5).

2.3 Proof of Theorem 1

With Lemma 2 and the observation from Section 2.2, we can now provide a proof of Theorem 1, applying the first and second moments method to the appropriate sets. In the proof, we use S_n to denote the walk defined by (7) and S_k to denote the sum of the first k summand in S_n.

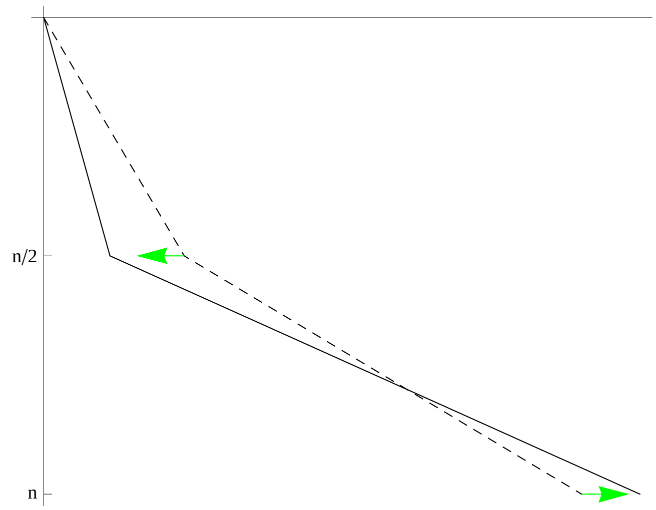


Figure 1: Dashed: maximum at time n of BRW starting from maximum at time $n/2$. Solid: maximum at time n of BRW starting from time 0.

Proof of Theorem 1. *Upper bound.* Let $a_n = \sqrt{(\sigma_1^2+\sigma_2^2)\log 2}n - \frac{\sqrt{\sigma_1^2+\sigma_2^2}}{4\sqrt{\log 2}}\log n$. Let $N_{1,n} = \sum_{v\in\mathbb{D}_n} 1_{\{S_v > a_n+y\}}$ be the number of particles at time n whose displacements are greater than $a_n + y$. Then

$$EN_{1,n} = 2^n P(S_n \geq a_n + y) \leq c_2 e^{-c_3 y}$$

where c_2 and c_3 are constants independent of n and the last inequality is due to the fact that $S_n \sim N(0, \frac{\sigma_1^2+\sigma_2^2}{2}n)$. So we have, by the Chebyshev's inequality,

$$P(M_n^{\uparrow} > a_n + y) = P(N_1 \geq 1) \leq EN_{1,n} \leq c_2 e^{-c_3 y}. \tag{16}$$

Therefore, this probability can be made as small as we wish by choosing a large y.

Lower bound. Consider the walks which are at $s_n \in I_n = [a_n, a_n + 1]$ at time n and follow $s_{k,n}(s_n)$, defined by (8), at intermediate times with fluctuation bounded by $f_{k,n}$, defined by (9). Let $I_{k,n}(x) = [s_{k,n}(x) - f_{k,n}, s_{k,n}(x) + f_{k,n}]$ be the 'admissible' interval at time k given $S_n = x$, and let

$$N_{2,n} = \sum_{v\in\mathbb{D}_n} 1_{\{S_v\in I_n, S_{v_k}\in I_{k,n}(S_v) \text{ for all } 0\leq k\leq n\}}$$

be the number of such walks. By Lemma 2,

$$\begin{aligned} EN_{2,n} &= 2^n P(S_n \in I_n, S_n(k) \in I_{k,n}(S_n) \text{ for all } 0 \leq k \leq n) \\ &= 2^n E(1_{\{S_n\in I_n\}} P(S_n(k) \in I_{k,n}(S_n) \text{ for all } 0 \leq k \leq n | S_n)) \\ &\geq 2^n C P(S_n \in I_n) \geq c_4. \end{aligned} \tag{17}$$

Next, we bound the second moment $EN_{2,n}^2$. By considering the location of any pair

$v_1, v_2 \in \mathbb{D}_n$ of particles at time n and at their common ancestor $v_1 \wedge v_2$, we have

$$
\begin{aligned}
EN_{2,n}^2 = E \sum_{v_1, v_2 \in \mathbb{D}_n} 1_{\{S_{v_i} \in I_n,\, S_{(v_i)^j} \in I_{j,n}(S_{(v_i)^j}) \text{ for all } 0 \le j \le n, i=1,2\}} \\
= \sum_{k=0}^{n} \sum_{\substack{v_1, v_2 \in \mathbb{D}_n \\ v_1 \wedge v_2 \in \mathbb{D}_k}} E 1_{\{S_{v_i} \in I_n,\, S_{(v_i)^j} \in I_{j,n}(S_{(v_i)^j}) \text{ for all } 0 \le j \le n, i=1,2\}} \\
\le \sum_{k=0}^{n} \sum_{\substack{v_1, v_2 \in \mathbb{D}_n \\ v_1 \wedge v_2 \in \mathbb{D}_k}} P(S_{v_1} \in I_n,\ S_{(v_1)^j} \in I_{j,n}(S_{(v_1)^j}) \text{ for all } 0 \le j \le n) \\
\cdot P(S_{v_2} - S_{v_1 \wedge v_2} \in [x - s_{k,n}(x) - f_{k,n}, x - s_{k,n}(x) + f_{k,n}], x \in I_n),
\end{aligned}
$$

where we use the independence between $S_{v_2} - S_{v_1 \wedge v_2}$ and $S_{(v_1)^j}$ in the last inequality. And the last expression (double sum) in the above display is

$$
\begin{aligned}
& \sum_{k=0}^{n} 2^{2n-k} P(S_n \in I_n, S_n(j) \in I_{j,n}(S_n) \text{ for all } 0 \le j \le n) \\
& \quad \cdot P(S_n - S_n(k) \in [x - s_{k,n}(x) - f_{k,n}, x - s_{k,n}(x) + f_{k,n}], x \in I_n) \\
\le \; & EN_{2,n} \sum_{k=0}^{n} 2^{n-k} P(S_n - S_n(k) \in [x - s_{k,n}(x) - f_{k,n}, x - s_{k,n}(x) + f_{k,n}], x \in I_n).
\end{aligned}
$$

The above probabilities can be estimated separately when $k \le n/2$ and $n/2 < k \le n$. For $k \le n/2$, $S_n - S_n(k) \sim N(0, \frac{n}{2}(\sigma_1^2 + \sigma_2^2) - k\sigma_1^2)$. Thus,

$$
\begin{aligned}
& P(S_n - S_n(k) \in [x - s_{k,n}(x) - f_{k,n}, x - s_{k,n}(x) + f_{k,n}], x \in I_n) \\
\le \; & 2f_{k,n} \frac{1}{\sqrt{\pi((\sigma_1^2 + \sigma_2^2)n - 2k\sigma_1^2)}} \exp\left(- \frac{\left((1 - \frac{2\sigma_1^2 k}{(\sigma_1^2 + \sigma_2^2)n}) a_n - f_{k,n} \right)^2}{(\sigma_1^2 + \sigma_2^2)n - 2k\sigma_1^2} \right) \\
\le \; & 2^{-n + \frac{2\sigma_1^2}{\sigma_1^2 + \sigma_2^2} k + o(k)}.
\end{aligned}
$$

For $n/2 < k \le n$, $S_n - S_n(k) \sim N(0, (n-k)\sigma_2^2)$. Thus,

$$
\begin{aligned}
& P(S_n - S_n(k) \in [x - s_{k,n}(x) - f_{k,n}, x - s_{k,n}(x) + f_{k,n}], x \in I_n) \\
\le \; & 2f_{k,n} \frac{1}{\sqrt{2\pi(n-k)\sigma_2^2}} \exp\left(- \frac{\left(\frac{2\sigma_2^2 (n-k)}{(\sigma_1^2 + \sigma_2^2)n} a_n - f_{k,n} \right)^2}{2(n-k)\sigma_2^2} \right) \\
\le \; & 2^{-\frac{2\sigma_2^2}{\sigma_1^2 + \sigma_2^2}(n-k) + o(n-k)}.
\end{aligned}
$$

Therefore,

$$
EN_{2,n}^2 \le EN_{2,n} \left(\sum_{k=0}^{n/2} 2^{\frac{\sigma_1^2 - \sigma_2^2}{\sigma_1^2 + \sigma_2^2} k + o(k)} + \sum_{k=n/2+1}^{n} 2^{\frac{\sigma_1^2 - \sigma_2^2}{\sigma_1^2 + \sigma_2^2}(n-k) + o(n-k)} \right) \le c_5 EN_{2,n}, \qquad (18)
$$

where $c_5 = 2\sum_{k=0}^{\infty} 2^{\frac{\sigma_1^2-\sigma_2^2}{\sigma_1^2+\sigma_2^2}k+o(k)}$. By the Cauchy-Schwartz inequality,

$$P(M_n^{\uparrow} \geq a_n) \geq P(N_{2,n} > 0) \geq \frac{(EN_{2,n})^2}{EN_{2,n}^2} \geq c_4/c_5 > 0. \tag{19}$$

The upper bound (16) and lower bound (19) imply that there exists a large enough constant y_0 such that

$$P(M_n^{\uparrow} \in [a_n, a_n + y_0]) \geq \frac{c_4}{2c_5} > 0.$$

Lemma 1 tells us that the sequence $\{M_n^{\uparrow} - \mathrm{Med}(M_n^{\uparrow})\}_n$ is tight, so $M_n^{\uparrow} = a_n + O(1)$ a.s.. That completes the proof. □

3 Decreasing Variances: $\sigma_1^2 > \sigma_2^2$

We will again separate the proof of Theorem 2 into two parts, the lower bound and the upper bound. Fortunately, we can apply (2) directly to get a lower bound so that we can avoid repeating the second moment argument. However, we do need to reproduce (the first moment argument) part of the proof of (2) in order to get an upper bound.

3.1 An Estimate for Brownian Bridge

Toward this end, we need the following analog of Bramson [4, Proposition 1']. The original proof in Bramson's used the Gaussian density and reflection principle of continuous time Brownian motion, which also hold for the discrete time version. The proof extends without much effort to yield the following estimate for the Brownian bridge $B_k - \frac{k}{n}B_n$, where B_n is a random walk with standard normal increments.

Lemma 3. *Let*

$$L(k) = \begin{cases} 0 & \text{if } s = 0, n, \\ 100\log k & \text{if } k = 1, \dots, n/2, \\ 100\log(n-k) & \text{if } k = n/2, \dots, n-1. \end{cases}$$

Then, there exists a constant C such that, for all $y > 0$,

$$P(B_n - \frac{k}{n}B_n \leq L(k) + y \text{ for } 0 \leq k \leq n) \leq \frac{C(1+y)^2}{n}.$$

The coefficient 100 before log is chosen large enough to be suitable for later use, and is not crucial in Lemma 3.

3.2 Proof of Theorem 2

Before proving the theorem, we discuss the equivalent optimization problems (13) and (14) under our current setting $\sigma_1^2 > \sigma_2^2$. It can be solved by employing the optimal curve

$$\phi(s) = \begin{cases} \sqrt{2\log 2}\sigma_1 s, & 0 \leq s \leq \frac{1}{2}, \\ \sqrt{2\log 2}\sigma_1 \frac{1}{2} + \sqrt{2\log 2}\sigma_2(s - \frac{1}{2}), & \frac{1}{2} \leq s \leq 1. \end{cases} \tag{20}$$

If we plot the curve $\phi(s)$ and the suboptimal curve leading to (6) as in Figure 2, these two curves coincide with each other up to order n. Figure 2 seems to indicate that the maximum at time n for the branching random walk starting from time 0 comes from the maximum at time $n/2$. As will be shown rigorously, if a particle at time $n/2$ is left significantly behind the maximum, its descendents will not be able to catch up by time n. The difference between Figure 1 and Figure 2 explains the difference in the logarithmic correction between $M_n^{\uparrow}$ and $M_n^{\downarrow}$.

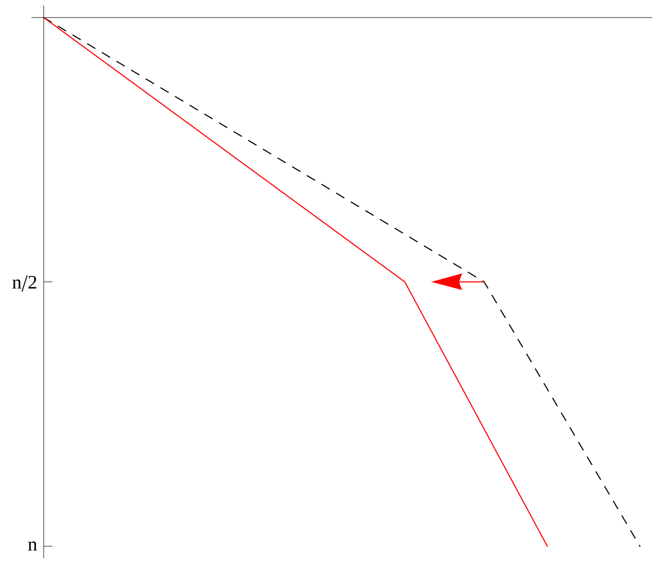


Figure 2: Dash: both the optimal path to the maximum at time n and the path leading to the maximum of BRW starting from the maximum at time $n/2$. Solid: the path to the maximal (rightmost) descendent of a particle at time $n/2$ that is significantly behind the maximum then.

Proof of Theorem 2. Lower Bound. For each $i = 1, 2$, the formula (2) implies that there exist y_i (possibly negative) such that, for branching random walk at time $n/2$ with variance σ_i^2,

$$P\left(M_{n/2} > \sqrt{2\log 2}\sigma_i \frac{n}{2} - \frac{3\sigma_i}{2\sqrt{2\log 2}} \log \frac{n}{2} + y_i\right) \geq \frac{1}{2}.$$

By considering a branching random walk starting from a particle at time $n/2$, whose location is greater than $\sqrt{2\log 2}\sigma_1 \frac{n}{2} - \frac{3\sigma_1}{2\sqrt{2\log 2}} \log \frac{n}{2} + y_1$, and applying the above display with $i = 1$ and 2,we know that

$$P\left(M_n^{\downarrow} > \frac{\sqrt{2\log 2}(\sigma_1+\sigma_2)}{2} n - \frac{3(\sigma_1+\sigma_2)}{2\sqrt{2\log 2}} \log \frac{n}{2} + y_1 + y_2\right) \geq \frac{1}{4}. \tag{21}$$

Upper Bound. We will use a first moment argument to prove that there exists a constant y (large enough) such that

$$P\left(M_n^{\downarrow} > \frac{\sqrt{2\log 2}(\sigma_1+\sigma_2)}{2} n - \frac{3(\sigma_1+\sigma_2)}{2\sqrt{2\log 2}} \log \frac{n}{2} + y\right) < \frac{1}{10}. \tag{22}$$

Similarly to the last argument in the proof of Theorem 1, the upper bound (22) and the lower bound (21), together with the tightness result from Lemma 1, prove Theorem 2. So it remains to show (22).

Toward this end, we define a polygonal line (piecewise linear curve) leading to $\frac{\sqrt{2\log 2}(\sigma_1+\sigma_2)}{2}n - \frac{3(\sigma_1+\sigma_2)}{2\sqrt{2\log 2}}\log\frac{n}{2}$ as follows: for $1 \le k \le n/2$,

$$M(k) = \frac{k}{n/2}(\sqrt{2\log 2}\sigma_1\frac{n}{2} - \frac{3\sigma_1}{2\sqrt{2\log 2}}\log\frac{n}{2});$$

and for $n/2+1 \le k \le n$,

$$M(k) = M(n/2) + \frac{k-n/2}{n/2}(\sqrt{2\log 2}\sigma_2\frac{n}{2} - \frac{3\sigma_2}{2\sqrt{2\log 2}}\log\frac{n}{2}).$$

Note that $\frac{k}{n}\log n \le \log k$ for $k \le n$. Also define

$$f(k) = \begin{cases} y & k = 0, \frac{n}{2}, n, \\ y + \frac{5\sigma_1}{2\sqrt{2\log 2}}\log k & 1 \le k \le n/4, \\ y + \frac{5\sigma_1}{2\sqrt{2\log 2}}\log(\frac{n}{2}-k) & \frac{n}{4} \le k \le \frac{n}{2}-1, \\ y + \frac{5\sigma_2}{2\sqrt{2\log 2}}\log(k-\frac{n}{2}) & \frac{n}{2}+1 \le k \le \frac{3n}{4}, \\ y + \frac{5\sigma_2}{2\sqrt{2\log 2}}\log(n-k) & \frac{3n}{4} \le k \le n-1. \end{cases}$$

We will use $f(k)$ to denote the allowed offset (deviation) from $M(k)$ in the following argument.

The probability on the left side of (22) is equal to

$$P(\exists v \in \mathbb{D}_n \text{ such that } S_v > M(n) + y).$$

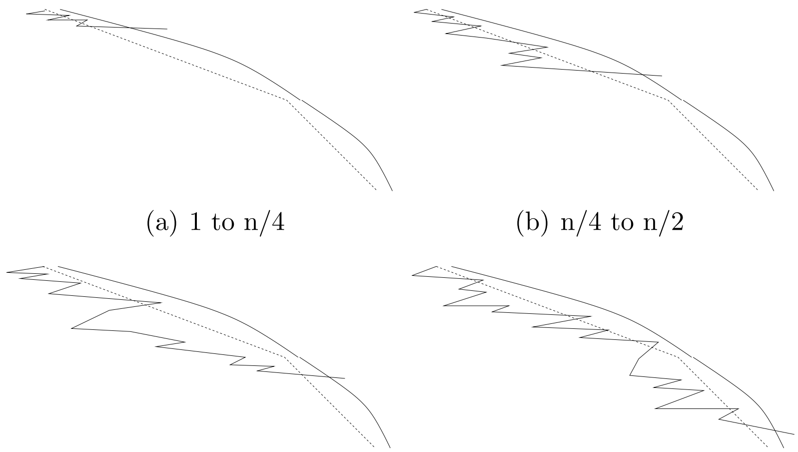
For each $v \in \mathbb{D}_n$, we define $\tau_v = \inf\{k : S_{v^k} > M(k) + f(k)\}$; then (22) is implied by

$$\sum_{k=1}^{n} P(\exists v \in \mathbb{D}_n \text{ such that } S_v > M(n) + y, \tau_v = k) < 1/10. \tag{23}$$

We will split the sum into four regimes: $[1, n/4]$, $[n/4, n/2]$, $[n/2, 3n/4]$ and $[3n/4, n]$, corresponding to the four parts of the definition of $f(k)$. The sum over each regime, corresponding to the events in the four pictures in Figure 3, can be made small. The first two are the discrete analog of the upper bound argument in Bramson [4]. We will present a complete proof for the first two cases, since the argument is not too long and the argument (not only the result) is used in the latter two cases.

(i). When $1 \le k \le n/4$, we have, by the Chebyshev's inequality,

$$\begin{aligned} & P(\exists v \in \mathbb{D}_n \text{ such that } S_v > M(n) + y, \tau_v = k) \\ \le\ & P(\exists v \in \mathbb{D}_k, \text{ such that } S_v > M(k) + f(k)) \le E\sum_{v\in\mathbb{D}_k} 1_{\{S_v > M(k)+f(k)\}}. \end{aligned}$$



(a) 1 to n/4 (b) n/4 to n/2

(c) n/2 to 3n/4 (d) 3n/4 to n

Figure 3: Four small probability events. Dash line: $M(k)$. Solid curve: $M(k)+f(k)$. Polygonal line: a random walk.

The above expectation is less than or equal to

$$\begin{aligned} \frac{C2^k}{\sqrt{k}} e^{-\frac{(M(k)+f(k))^2}{2\sigma_1^2}} &\leq \frac{C2^k}{\sqrt{k}} \exp\left(- \frac{\left(\sqrt{2\log 2}\sigma_1 k + \frac{\sigma_1}{\sqrt{2\log 2}} \log k + y\right)^2}{2k\sigma_1^2} \right) \\ &\leq Ck^{-3/2} e^{-\frac{\sqrt{2\log 2}}{\sigma_1} y}. \end{aligned} \tag{24}$$

Summing these upper bounds over $k \in [1, n/4]$, we obtain that

$$\sum_{k=1}^{n/4} P(\exists v \in \mathbb{D}_n \text{ such that } S_v > M(n) + y, \tau_v = k) \leq C e^{-\frac{\sqrt{2\log 2}}{\sigma_1} y} \sum_{k=1}^{\infty} k^{-3/2}. \tag{25}$$

The right side of the above inequality can be made as small as we wish, say at most $\frac{1}{100}$, by choosing y large enough.

(ii). When $n/4 \leq k \leq n/2$, we again have, by Chebyshev's inequality,

$$\begin{aligned} & P(\exists v \in \mathbb{D}_n \text{ such that } S_v > M(n) + y, \tau_v = k) \\ \leq\ & P(\exists v \in \mathbb{D}_k, \text{ such that } S_v > M(k) + f(k), \text{ and } S_{v^i} \leq M(i) + f(i) \text{ for } 1 \leq i \leq k) \\ \leq\ & E \sum_{v \in \mathbb{D}_k} 1_{\{S_v > M(k)+f(k), \text{ and } S_{v^i} \leq M(i)+f(i) \text{ for } 1 \leq i < k\}}. \end{aligned}$$

Letting S_k be a copy of the random walks before time $n/2$, then the above expectation is equal to

$$\begin{aligned} & 2^k P(S_k > M(k) + f(k), \text{ and } S_i \leq M(i) + f(i) \text{ for } 1 \leq i < k) \\ \leq\ & 2^k P(S_k > M(k) + f(k), \text{ and } \frac{1}{\sigma_1}(S_i - \frac{i}{k} S_k) \leq \frac{1}{\sigma_1}(f(i) - \frac{i}{k} f(k)) \text{ for } 1 \leq i \leq k). \end{aligned} \tag{26}$$

$\frac{1}{\sigma_1}(S_i - \frac{i}{k}S_k)$ is a discrete Brownian bridge and is independent of S_k. Because of this independence, the above quantity is less than or equal to

$$2^k P(S_k > M(k) + f(k)) \cdot P(\frac{1}{\sigma_1}(S_i - \frac{i}{k}S_k) \leq \frac{1}{\sigma_1}(f(i) - \frac{i}{k}f(k)) \text{ for } 1 \leq i < k).$$

The first probability can be estimated similarly to (24),

$$\begin{aligned} & P(S_k > M(k) + f(k)) \\ \leq\ & \frac{C}{\sqrt{k}} \exp\left(- \frac{\left(\sqrt{2\log 2}\sigma_1 k - \frac{3\sigma_1}{2\sqrt{2\log 2}}\log k + \frac{5\sigma_1}{2\sqrt{2\log 2}}\log(\frac{n}{2} - k) + y\right)^2}{2k\sigma_1^2}\right) \\ \leq\ & C 2^{-k} k (\frac{n}{2} - k)^{-5/2} e^{-\frac{\sqrt{2\log 2}}{\sigma_1}y}. \end{aligned} \tag{27}$$

To estimate the second probability, we first estimate $\frac{1}{\sigma_1}(f(i) - \frac{i}{k}f(k))$. It is less than or equal to $\frac{1}{\sigma_1}f(i) = \frac{y}{\sigma_1} + \frac{5}{2\sqrt{2\log 2}}\log i$ for $i \leq k/2 < n/4$, and, for $k/2 \leq i < k$, it is less than or equal to

$$\begin{aligned} & \frac{5}{2\sqrt{2\log 2}}\log(n/2 - i) - \frac{i}{k}\frac{5}{2\sqrt{2\log 2}}\log(n/2 - k) + \frac{y}{\sigma_1}(1 - \frac{i}{k}) \\ =\ & \frac{5}{2\sqrt{2\log 2}}\left(\log(n/2 - i) - \log(n/2 - k) + \frac{k-i}{k}\log(n/2 - k)\right) + \frac{y}{\sigma_1}(1 - \frac{i}{k}) \\ \leq\ & \frac{5}{2\sqrt{2\log 2}}\left(\log(k - i) + \frac{k-i}{k}\log k\right) + \frac{y}{\sigma_1} \leq 100\log(k - i) + \frac{y}{\sigma_1}. \end{aligned}$$

Therefore, applying Lemma 3, we have

$$\begin{aligned} & P\left(\frac{1}{\sigma_1}(S_i - \frac{i}{k}S_k) \leq \frac{1}{\sigma_1}(f(i) - \frac{i}{k}f(k)) \text{ for } 1 \leq i \leq k\right) \\ \leq\ & P\left(\frac{1}{\sigma_1}(S_i - \frac{i}{k}S_k) \leq 100\log i + \frac{y}{\sigma_1} \text{ for } 1 \leq i \leq k/2, \text{ and } \frac{1}{\sigma_1}(S_i - \frac{i}{k}S_k) \leq \right. \\ & \left. 100\log(k - i) + \frac{y}{\sigma_1} \text{ for } k/2 \leq i \leq k\right) \leq C(1 + y)^2/k, \end{aligned} \tag{28}$$

where C is independent of n, k and y.

By all the above estimates (26), (27) and (28),

$$\sum_{k=n/4}^{n/2} P(\exists v \in \mathbb{D}_n \text{ such that } S_v > M(n) + y, \tau_v = k) \leq C(1 + y)^2 e^{-\frac{\sqrt{2\log 2}}{\sigma_1}y}\sum_{k=1}^{\infty} k^{-5/2}. \tag{29}$$

This can again be made as small as we wish, say at most $\frac{1}{100}$, by choosing y large enough.

(iii). When $n/2 \leq k \leq 3n/4$, we have

$$\begin{aligned} & P(\exists v \in \mathbb{D}_n \text{ such that } S_v > M(n) + y, \tau_v = k) \\ \leq\ & P(\exists v \in \mathbb{D}_k \text{ such that } S_v > M(k) + f(k) \text{ and } S_{v^i} \leq M(i) + f(i) \text{ for } 1 \leq i \leq n/2) \\ \leq\ & E\sum_{v \in \mathbb{D}_k} 1_{\{S_v > M(k) + f(k), \text{ and } S_{v^i} \leq M(i) + f(i) \text{ for } 1 \leq i < n/2\}}. \end{aligned}$$

The above expectation is, by conditioning on $\{S_{v^{n/2}} = M(n) + x\}$,

$$2^k \int_{-\infty}^{y} P(S'_{k-n/2} > M(k) - M(n/2) + f(k) - x) \cdot$$
$$\cdot P(S_i - \frac{i}{n/2} S_{n/2} \leq f(i) - \frac{i}{k} x \text{ for } 1 \leq i < n/2) \cdot$$
$$\cdot p_{S_{n/2}}(M(n/2) + x) dx, \tag{30}$$

where S and S' are two copies of the random walks before and after time $n/2$, respectively, and $p_{S_{n/2}}(x)$ is the density of $S_{n/2} \sim N(0, \frac{\sigma_1^2 n}{2})$.

We then estimate the three factors of the integrand separately. The first one, which is similar to (24), is bounded above by

$$P(S'_{k-n/2} > M(k) - M(n/2) + f(k) - x) \leq \frac{C}{\sqrt{k - n/2}} e^{-\frac{(M(k)-M(n/2)+f(k)-x)^2}{2(k-n/2)\sigma_2^2}}$$
$$\leq \; C 2^{-(k-n/2)} (k - \frac{n}{2})^{-3/2} e^{-\frac{\sqrt{2\log 2}}{\sigma_2}(y-x)}.$$

The second one, which is similar to (28), is estimated using Lemma 3,

$$P(S_i - \frac{i}{n/2} S_{n/2} \leq f(i) - \frac{i}{k} x \text{ for } 1 \leq i < n/2) \leq C(1 + 2y - x)^2 / n. \tag{31}$$

The third one is simply the normal density

$$p_{S_{n/2}}(M(n/2) + x) = \frac{C}{\sqrt{n}} e^{-\frac{(M(n/2)+x)^2}{n\sigma_1^2}} \leq C 2^{-n/2} n e^{-\frac{\sqrt{2\log 2}}{\sigma_1} x}. \tag{32}$$

Therefore, the integral term (30) is no more than

$$C(k - n/2)^{-3/2} e^{-\frac{\sqrt{2\log 2}}{\sigma_2} y} \int_{-\infty}^{y} (1 + 2y - x)^2 e^{(\frac{\sqrt{2\log 2}}{\sigma_2} - \frac{\sqrt{2\log 2}}{\sigma_1}) x} dx,$$

which is less than or equal to $C(1+y)^2 e^{-\frac{\sqrt{2\log 2}}{\sigma_1} y} (k - n/2)^{-3/2}$ since $\sigma_2 < \sigma_1$.

Summing these upper bounds together, we obtain that

$$\sum_{k=n/2}^{3n/4} P(\exists v \in \mathbb{D}_n \text{ such that } S_v > M(n) + y, \tau_v = k) \leq C(1+y)^2 e^{-\frac{\sqrt{2\log 2}}{\sigma_1} y} \sum_{k=1}^{\infty} k^{-3/2}. \tag{33}$$

This can again be made as small as we wish, say at most $\frac{1}{100}$, by choosing y large enough.

(iv). When $3n/4 < k \leq n$, we have

$$\begin{aligned} & P(\exists v \in \mathbb{D}_n \text{ such that } S_v > M(n) + y, \tau_v = k) \\ \leq \; & P(\exists v \in \mathbb{D}_k \text{ such that } S_v > M(k) + f(k), \text{ and } S_{v^i} \leq M(i) + f(i) \text{ for } 1 \leq i < k) \\ \leq \; & E \sum_{v \in \mathbb{D}_k} 1_{\{S_v > M(k) + f(k), \text{ and } S_{v^i} \leq M(i) + f(i), \text{ for } 1 \leq i < k\}}. \end{aligned}$$

The above expectation is, by conditioning on $\{S_{v^{n/2}} = M(n) + x\}$,

$$2^k \int_{-\infty}^{y} P(S'_{k-n/2} > M(k) - M(n/2) + f(k) - x,$$
$$S'_i < M(i) - M(n/2) + f(i) - x, \text{ for } n/2 < i \leq k)$$
$$\cdot P(S_i - \frac{i}{n/2} S_{n/2} \leq f(i) - \frac{i}{k} x \text{ for } 1 \leq i < n/2) \cdot p_{S_{n/2}}(M(n/2) + x) dx$$

where S and S' are copies of the random walks before and after time $n/2$, respectively.

The second and third probabilities in the integral are already estimated in (31) and (32). It remains to bound the first probability. Similar to (26), it is bounded above by

$$P\left(S'_{k-n/2} > M(k) - M(n/2) + f(k) - x, S'_i < M(i) - M(n/2) + f(i) - x,\right.$$
$$\left.\text{for } n/2 < i \leq k\right) \leq C(1 + 2y - x)^2 e^{-\frac{\sqrt{2\log 2}}{\sigma_2}(2y-x)} (n-k)^{-5/2}.$$

With these estimates, we obtain in this case, in the same way as in (iii), that

$$\sum_{k=3n/4}^{n} P(\exists v \in \mathbb{D}_n \text{ such that } S_v > M(n) + y, \tau_v = k) \leq C(1+y)^2 e^{-\frac{\sqrt{2\log 2}}{\sigma_1} y} \sum_{k=1}^{\infty} k^{-5/2}. \quad (34)$$

This can again be made as small as we wish, say at most $\frac{1}{100}$, by choosing y large enough.

Summing (25), (29), (33) and (34), then (23) and thus (22) follow. This concludes the proof of Theorem 2. □

4 Further Remarks

We state several immediate generalization and open questions related to binary branching random walks in time inhomogeneous environments where the diffusivity of the particles takes more than two distinct values as a function of time and changes macroscopically.

Results involving finitely many monotone variances can be obtained similarly to the results on two variances in the previous sections. Specifically, let $k \geq 2$ (constant) be the number of inhomogeneities, $\{\sigma_i^2 > 0 : i = 1, \ldots, k\}$ be the set of variances and $\{t_i > 0 : i = 1, \ldots, k\}$, satisfying $\sum_{i=1}^k t_i = 1$, denote the portions of time when σ_i^2 governs the diffusivity. Consider binary branching random walk up to time n, where the increments over the time interval $[\sum_{i=1}^{j-1} t_i n, \sum_{i=1}^{j} t_i n)$ are $N(0, \sigma_j^2)$ for $1 \leq j \leq k$. When $\sigma_1^2 < \sigma_2^2 < \cdots < \sigma_k^2$ are strictly increasing, by an argument similar to that in Section 2, the maximal displacement at time n, which behaves asymptotically like the maximum for independent random walks with effective variance $\sum_{i=1}^k t_i \sigma_i^2$, is

$$\sqrt{2(\log 2) \sum_{i=1}^{k} t_i \sigma_i^2 n} - \frac{1}{2} \frac{\sqrt{\sum_{i=1}^k t_i \sigma_i^2}}{\sqrt{2\log 2}} \log n + O_P(1).$$

When $\sigma_1^2 > \sigma_2^2 > \cdots > \sigma_k^2$ are strictly decreasing, by an argument similar to that in Section 3, the maximal displacement at time n, which behaves like the sub-maximum chosen by the

previous greedy strategy (see (6)), is

$$\sqrt{2\log 2}(\sum_{i=1}^{k} t_i\sigma_i)n - \frac{3}{2}(\sum_{i=1}^{k} \frac{\sigma_i}{\sqrt{2\log 2}})\log n + O_P(1).$$

Results on other inhomogeneous environments are open and are subjects of further study. We only discuss some of the non rigorous intuition in the rest of this section.

In the finitely many variances case, when $\{\sigma_i^2 : i = 1, \ldots, k\}$ are not monotone in i, the analysis of maximal displacement could be case-by-case and a mixture of the previous monotone cases. The leading order term is surely a result of the optimization problem (12) from the large deviation. But, the second order term may depend on the fluctuation constraints of the path leading to the maximum, as in the monotone case. One could probably find hints on the fluctuation from the optimal curve solving (12). In some segments, the path may behave like Brownian bridge (as in the decreasing variances case), and in some segments, the path may behave like a random walk (as in the increasing variances case).

In the case where the number of different variances increases as the time n increases, analysis seems more challenging. A special case is when the variances are decreasing, for example, at time $0 \leq i \leq n$ the increment of the walk is $N(0, \sigma_{i,n}^2)$ with $\sigma_{i,n}^2 = 2 - i/n$. The heuristics (from the finitely many decreasing variances case) seem to indicate that the path leading to the maximum at time n cannot be left 'significantly' behind the maxima at all intermediate levels. This path is a 'rightmost' path. From the intuition of [11], if the allowed fluctuation is of order n^α ($\alpha < 1/2$), then the correction term is of order $n^{1-2\alpha}$, instead of $\log n$ in (1). However, the allowed fluctuation from the intermediate maxima, implicitly imposed by the variances, becomes complicated as the difference between the consecutive variances decreases to zero. A good understanding of this fluctuation may be a key to finding the correction term.

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